UNSTEADY ENERGY TRANSFER FROM A RADIATING SPHERE IN A MEDIUM WITH MOLECULAR HEAT CONDUCTION

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O. V. VOINOV, A. M. GOLOVIN and A. G. PETROV
(Moscow)

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The temperature distribution around a radiating sphere in a homogeneous gas medium is considered with allowance for molecular heat transfer. Local thermodynamic equilibrium is assumed. The temperature is determined from an equation derived under the assumption that the photon path length $1/\alpha$ is much larger than the radius $\tilde{a}$ of the sphere. The general solution of the linearized energy transfer equation is written out. The behavior of the Green function for small and large times is investigated.

The temperature distribution in the range $\alpha r \lesssim \sqrt{\varkappa/\varkappa_r}$ is investigated in the particular case $\varkappa \ll \varkappa_r$, $\sqrt{\varkappa/\varkappa_r} \gg \alpha a$ (where $\varkappa$, $\varkappa_r$ are the coefficients of molecular and radiant thermal conductivity). The characteristic temperature relaxation time is determined.

1. The basic equations. Energy propagation from a radiating sphere produces nonuniform heating of the gas medium and radial gas motion. With a sufficiently small temperature drop between the surface of the sphere (at the temperature T_a) and points far away from it (at the temperature T_∞) the velocities v of gas motion are negligibly small as compared with the velocity of sound, so that the pressure has a chance to smooth itself out and heating of the gas proceeds at constant pressure.

The equation of energy transfer in a gas is [1]

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{v}\nabla T\right) + \operatorname{div}(\mathbf{S}_r - \varkappa\nabla T) = 0 \qquad (1.1)$$

Here ρ is the density of the gas medium; c_p is the specific heat at constant pressure; t is the time; S_r is the radiation flux density; $\varkappa$ is the molecular thermal conductivity coefficient.

We have omitted the energy flux associated with internal friction processes from Eq. (1.1). This flux is quadratic in velocity and therefore of a higher order of smallness over the temperature drop.

We must now apply the continuity equations [2] for the gas density ρ and the radiation energy density U, namely

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\mathbf{v} = 0, \qquad \operatorname{div} \mathbf{S}_r = \alpha c (U - U_p)\left(U_p = \frac{4\sigma}{c} T^4\right) \qquad (1.2)$$

Here c is the velocity of light, U_p is the energy density of the equilibrium radiation, and σ is the Stefan-Boltzmann constant.

A gas medium is described by the equation of state of an ideal gas which together with the condition of constant pressure during heating of the gas implies that $\rho T = \rho_\infty T_\infty$ (where ρ_∞ is the gas density far away from the drop).

Let us assume that the radiation path length $1/\alpha$ is much larger than the radius a of the sphere. Overlooking the dependence of α on temperature and density (as in [3]), we obtain the following equation describing the temperature distribution around the sphere:

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{v}\nabla T\right) - \varkappa\nabla T - 4\alpha\sigma T^4 + 2\frac{\alpha\sigma}{r}\int_a^\infty r' T^4(r') \int_{|r-r'|}^{r+r'} e^{-\alpha u}\frac{du}{u}\, dr' + \qquad (1.3)$$

$$+ 2\alpha\varepsilon\sigma\left[T_a^4 - \alpha\int_a^\infty T^4(r)\, e^{-\alpha r}\, dr\right]\left[1 - \left(1 - \frac{a^2}{r^2}\right)^{1/2}\right] e^{-\alpha r} = 0$$

Here ε is the effective emissivity in the sphere immersed in the gas, and r is the distance from the center of the sphere.

Let us introduce the new function φ,

$$T^4 = T_\infty^4 (1 + \varphi), \qquad T_a^4 = T_\infty^4 (1 + \varphi_a) \tag{1.4}$$

Setting $|\varphi| \ll 1$, we can linearize Eq. (1.3),

$$\mu^2 \frac{\partial^2 r\varphi}{\partial r^2} - \frac{\partial r\varphi}{\partial \tau} = r\varphi - \frac{\alpha}{2}\int_a^\infty r'\varphi(r', \tau) \int_{|r-r'|}^{r+r'} e^{-\alpha u}\frac{du}{u}\, dr' - A\left(r - \sqrt{r^2 - a^2}\right) e^{-\alpha r}\eta(r - a) \tag{1.5}$$

Here

$$\mu^2 = \frac{\varkappa}{16\alpha\sigma T_\infty^3}, \qquad \tau = \frac{\chi t}{\mu^2}, \qquad \chi = \frac{\varkappa}{\rho c_-}$$

$$A = \frac{\varepsilon}{2}\left(\varphi_a - \alpha\int_a^\infty \varphi e^{-\alpha r} dr\right), \qquad \eta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

We have omitted the convective term from Eq. (1.5), since v is a quantity of the order of $\partial\varphi / \partial t$ in accordance with the linearized first equation of (1.2), which means in turn that $\mathbf{rv}\nabla\varphi$ is a small quantity of the order of φ^2.

2. The general form of the solution of the linearized equation.

If we complement the definition of $\varphi(r)$ for $r < 0$ in even fashion, i.e. by stipulating that $\varphi(-r) = \varphi(r)$, then, neglecting quantities of the order αa and $\alpha a \ln \alpha a$, we can replace (1.5) by the equation [3]

$$\mu^2 \frac{\partial^2 r\varphi}{\partial r^2} - \frac{\partial r\varphi}{\partial \tau} = r\varphi - \frac{\alpha}{2}\int_0^\infty r'\varphi(r', \tau) E_1(\alpha|r - r'|)\, dr' - \tag{2.1}$$

$$- Ar\left[1 - \left(1 - \frac{a^2}{r^2}\right)^{1/2}\right] e^{-\alpha|r|}\eta(r^2 - a^2) \qquad \left(E_1(x) = \int_x^\infty e^{-u}\frac{du}{u}\right)$$

under the arbitrary initial and boundary conditions

$$\begin{aligned} &\varphi(r, \tau) \to \varphi_a(\tau) \quad \text{for} \quad r \to a + 0, \ \tau \neq 0 \\ &\varphi(r, \tau) \to 0 \quad \text{for} \quad r \to \infty \\ &\varphi(r, \tau) \to \varphi_0(r) \quad \text{for} \quad \tau \to 0, \ r \neq a \end{aligned} \tag{2.2}$$

It is also convenient to assume that the boundary condition for $r = +0$ has been given. Let $r\varphi(r, \tau) \to aB(\tau)$ for $r \to +0$, $\tau > 0$. If the characteristic time of variation of the boundary value of the temperature $\varphi_a(\tau)$ is large as compared with a^2/χ, i.e. with the time of establishment of the temperature in the boundary layer due to molecular heat conduction, then (as is done in [3] in considering a steadystate temperature distribution) we can assume that $B(\tau) = \varphi_a(\tau)$ for $\mu \gg a$. We shall confine ourselves to this case.

Let us solve Eq. (2.1) with the aid of the Fourier and Laplace transformations

$$\Phi(k, p) = \int_0^\infty e^{-p\tau} d\tau \int_{-\infty}^\infty r\varphi(r, \tau)\, e^{-ikr}\, dr, \qquad \Phi_0(k, p) = \frac{1}{p}\int_{-\infty}^\infty r\varphi_0(r)\, e^{-ikr} dr \tag{2.3}$$

$$A^*(p) = \int_0^\infty e^{-p\tau} A(\tau)\, d\tau, \qquad B^*(p) = \int_0^\infty e^{-p\tau}\varphi_a(\tau)\, d\tau$$

The transform of Eq. (2.1) is of the form

$$-\mu^2 k^2 \Phi - 2\mu^2 ikaB^* - p(\Phi - \Phi_0) = \Phi - A^* F - \frac{\alpha}{k}\operatorname{arc\,tg}\frac{k}{\alpha} \tag{2.4}$$

$$F = -ia^2 \operatorname{arc\,tg}(k/\alpha)$$

This implies that

$$\Phi - \Phi_\infty = \frac{p(\Phi_0 - \Phi_\infty)}{1 + p + \mu^2 k^2 - (\alpha / k) \operatorname{arc\,tg}(k/\alpha)} \tag{2.5}$$

$$\Phi_\infty = \frac{A^* F - 2\mu^2 ika B^*}{1 + \mu^2 k^2 - (\alpha / k) \operatorname{arc\,tg}(k/\alpha)}$$

If φ_a is independent of τ, then Φ_∞ is the transform of the steadystate temperature distribution obtained in [3]. In the general case $\Phi_\infty(k, p)$ corresponds to

$$r\varphi_\infty(r, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(AF - 2\mu^2 ika\varphi_a) e^{ikr} dk}{1 + \mu^2 k^2 - (\alpha / k) \operatorname{arc\,tg}(k/\alpha)} \tag{2.6}$$

Making use of the familiar convolution theorems for Fourier and Laplace transforms, we readily obtain the following expressions from (2.5):

$$r(\varphi - \varphi_\infty) = \int_{-\infty}^{\infty} r' \left[\varphi_0(r') G(r - r', \tau) - \frac{d}{d\tau} \int_0^\tau G(r - r', \tau - \tau') \varphi_\infty(r', \tau') d\tau'\right] dr' \tag{2.7}$$

$$G(r, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[ikr - \left(1 + \mu^2 k^2 - \frac{\alpha}{k} \operatorname{arc\,tg} \frac{k}{\alpha}\right)\tau\right] dk$$

3. The Green function. For large times $\tau \gg 1$ the major contribution to the integral which defines the Green function $G(r, \tau)$ is associated with small k,

$$G(r, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikr - \mu_*^2 k^2 \tau) dk =$$

$$= \frac{1}{2\mu_* \sqrt{\pi\tau}} \exp\left(-\frac{r^2}{4\mu_*^2 \tau}\right) \qquad \left(\mu_*^2 = \mu^2 + \frac{1}{3\alpha^2}\right) \tag{3.1}$$

This result agrees with that of [4], whose author considers a similar problem for the plane-parallel case with the radiation transfer equation in the Schwarzschild approximation, i. e. with the introduction of unilateral fluxes.

Green function (3.1) is the Green function of the ordinary heat conduction equation with the coefficient of thermal diffusivity equal to the sum of coefficients of molecular and radiant thermal diffusivity.

For small times $\tau \ll 1$ we can expand $\exp\{-\tau[1 - (\alpha / k) \operatorname{arctg}(k / \alpha)]\}$ in a series, retaining only the term of the first order of smallness. The Green function now becomes

$$G(r, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikr - \mu^2 k^2 \tau)\left(1 - \tau + \frac{\alpha\tau}{k} \operatorname{arc\,tg} \frac{k}{\alpha}\right) dk = \tag{3.2}$$

$$= \frac{1 - \tau}{2\mu \sqrt{\pi\tau}} \exp\left(-\frac{r^2}{4\mu^2 \tau}\right) + \frac{\tau}{2\pi} \int_{-\infty}^{\infty} \exp(ikr - \mu^2 k^2 \tau) \frac{\alpha}{k} \operatorname{arc\,tg} \frac{k}{\alpha} dk$$

For a small $\tau \ll \min(1, 1 / \mu^2\alpha^2)$ we assume that $\exp(-\mu^2 k^2 \tau) \approx 1$, since the major contribution to the integral is associated with the values $k \sim \alpha$.

Hence, we can write

$$G(r, \tau) = \frac{1 - \tau}{2\mu \sqrt{\pi\tau}} \exp\left(-\frac{r^2}{4\mu^2 \tau}\right) + \frac{\alpha\tau}{2} E_1(\alpha \mid r \mid) \tag{3.3}$$

This result differs from that obtained earlier for the plane-parallel case [4] in the fact that the exponential has been replaced by an integral exponential.

It is still true, however, that the effect of molecular heat conduction is substantial in the range of r values of the order of $\mu\sqrt{\tau}$.

If $\mu^2\alpha^2 \gg 1$, then $\tau \ll 1 / \mu^2\alpha^2$, and the region under consideration corresponds to $r \ll 1 / \alpha$. In the case $\mu^2\alpha^2 \ll 1$ formula (3.3) is valid for $\tau \ll 1$ so that the domain where the effect of molecular heat conduction is substantial is restricted by the condition $r \ll \mu\sqrt{r}$.

4. A particular solution. Let $\varphi_0(r) \equiv 0$, $\varphi_a(\tau) = \varphi_a \eta(\tau)$. Then, by (2.7) and (2.6),

$$r(\varphi_\infty - \varphi) = \int_0^\infty r'\varphi_\infty(r')\,[G(r - r', \tau) - G(r + r', \tau)]\,dr' \tag{4.1}$$

As is shown in [3], the steadystate temperature distribution in the region $r \lesssim \mu$ for $\mu^2\alpha^2 \ll 1$, $\mu \gg a$ is of the form

$$r\varphi_\infty(r) = a\varphi_a \exp(-r/\mu) \tag{4.2}$$

Thus, for small times $\tau \ll 1$ in the region $r \lesssim \mu$ we have

$$r(\varphi_\infty - \varphi) = a\varphi_a \int_0^\infty \frac{1-\tau}{2\mu\sqrt{\pi\tau}} \left\{\exp\left[-\frac{(r-r')^2}{4\mu^2\tau}\right] - \exp\left[-\frac{(r+r)^2}{4\mu^2\tau}\right]\right\} \exp\left(-\frac{r'}{\mu}\right) dr' =$$

$$= \frac{a\varphi_a}{2}\left[\exp\left(-\frac{r}{\mu}\right)\operatorname{erfc}\left(\sqrt{\tau} - \frac{r}{2\mu\sqrt{\tau}}\right) - \exp\left(\frac{r}{\mu}\right)\operatorname{erfc}\left(\sqrt{\tau} + \frac{r}{2\mu\sqrt{\tau}}\right)\right] \tag{4.3}$$

$$\left(\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt\right)$$

For large times $\tau \gg 1$ in the same region we have

$$r(\varphi_\infty - \varphi) = \frac{a\varphi_a}{2}\left[\exp\left(\tau - \frac{r}{\mu}\right)\operatorname{erfc}\left(\frac{\mu_*\sqrt{\tau}}{\mu} - \frac{r}{2\mu_*\sqrt{\tau}}\right) - \right. \tag{4.4}$$

$$\left. - \exp\left(\tau + \frac{r}{\mu}\right)\operatorname{erfc}\left(\frac{\mu_*\sqrt{\tau}}{\mu} + \frac{r}{2\mu_*\sqrt{\tau}}\right)\right]$$

Formulas (4.3) and (4.4) imply that the characteristic temperature relaxation time, and therefore the relaxation time of the heat flux due to molecular heat conduction is r^2 / χ for $t \ll \mu^2 / \chi$ $(\tau \ll 1)$, and $\mu^4\alpha^2 / \chi \sim \mu^2 / \chi_r$ for $t \gg \mu^2 / \chi$ (where $\chi_r = \chi\varkappa_r/\varkappa$ is the radiant thermal conductivity and $\varkappa_r$ is the coefficient of radiant thermal conductivity. The condition $\chi_r \gg \chi$ means that temperature relaxation in the region $r \lesssim \mu$ occurs in times of the order of μ^2 / χ. This means that radiation does not affect temperature relaxation in the range $\alpha r \lesssim \sqrt{\varkappa / \varkappa_r}$.

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